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LETTER TO THE EDITOR

The non-adiabatic Berry phase in two dimensions: the Calogero model trapped in a time-dependent external field

Jian-Sheng Wu†, Zhi-Ming Bai† and Mo-Lin Ge†‡

† Theoretical Physics Division, Nankai Institute of Mathematics, Nankai University, Tianjin 300071, People's Republic of China

‡ Center for Advanced Study, Tsinghua University, Beijing 100084, People's Republic of China

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Abstract. The two-dimensional Calogero model in a time-independent external magnetic field is extended to the time-dependent case which can be solved exactly. Both the invariant operator and the eigenstate are obtained. For the periodical time-dependent case, the non-adiabatic Berry phase is also presented.

1. Introduction

Multi-body problems have always attracted much interest in the field of atomic and nuclear physics. A celebrated example of a solvable model is the well known Calogero model in one or two dimensions [1–5]. In these models, not only do N particles interact with each other through a two-body potential which varies as the inverse square of the distance between two particles, but they are also trapped in a harmonic potential well. If the charged particle system described by the Calogero model in two dimensions is placed in a static magnetic field which does not vary with space and time [6], it is shown that the system is analytically solvable, especially for the single-body problem [7]. As we known, these solvable multi-body models are all time independent. An interesting problem is that the magnetic field acting on the Calogero model varies with time. It is shown that this system is also exactly solvable and the non-adiabatic Berry phase can be obtained if the magnetic field is periodic in time.

This paper is arranged as follows. In section 2 the general method for solving the time-dependent system [8] is given. In section 3 the method is applied to the two-dimensional Calogero model which is acted upon by a time-dependent magnetic field along the z-axis. The invariant of the system and the non-adiabatic Berry phase are obtained. Some conclusions are given in section 4.

2. The general method for solving a time-dependent quantum system

Consider the time-dependent Schrödinger equation ($\hbar = 1$)

$$\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}|\psi(t)\rangle = H(t)|\psi(t)\rangle. \qquad (1)$$

In order to solve this equation [2], a unitary transformation for the wavefunction is made

$$|\psi(t)\rangle' = U^{-1}(t)|\psi(t)\rangle. \qquad (2)$$

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It is obvious that if the Hamiltonian makes the following transformation:

$$U^{-1}(t)H(t)U(t) - \mathrm{i}U^{-1}(t)\frac{\partial U(t)}{\partial t} \equiv H(t)' \tag{3}$$

the Schrödinger equation maintains its original form, i.e.

$$\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}|\psi(t)\rangle' = H(t)'|\psi(t)\rangle'. \tag{4}$$

Our aim is to choose several appropriate unitary transformations to make the time-dependent Hamiltonian take the general Hill–Floquet–Bloch (HFB) form, i.e. the time-dependent Hamiltonian H' is transformed into the form

$$H(t)' = \eta(t)H_0' \tag{5}$$

where H_0' is a time-independent Hamiltonian, but $\eta(t)$ is a time-dependent function. In this case the solution for $|\psi\rangle'$ takes the following form:

$$|\psi(t)\rangle' = \exp\left[-\mathrm{i}H_0'\int_{t_0}^{t}\eta(s)\,\mathrm{d}s\right]|\phi\rangle \tag{6}$$

where $|\phi\rangle$ can be expressed by the eigenstates of the time-independent Hamiltonian H_0' , i.e. $H_0|\phi_n\rangle = E_n|\phi_n\rangle$ and $|\phi\rangle = \sum a_n|\phi_n\rangle$, in which a_n are some constants. On account of the transformations in equations (2) and (6) the solution for the Schrödinger equation (1) is

$$|\psi(t)\rangle = U(t)\sum a_n \exp\left[-\mathrm{i}E_n\int_{t_0}^{t}\eta(s)\,\mathrm{d}s\right]|\phi_n\rangle. \tag{7}$$

With the help of equation (3), we find that $I(t) = U(t)H_0U^{-1}(t)$ is the invariant of the system, i.e. $I(t)$ satisfies the equation $\frac{\mathrm{d}I}{\mathrm{d}t} = \frac{\partial I}{\partial t} + \mathrm{i}[H, I] = 0$. It is easy to prove that the eigenfunction of $I(t)$ is

$$|\lambda_n(t)\rangle = U(t)|\phi_n\rangle \tag{8}$$

and its eigenvalues are the same as those of H_0. In order to give the non-adiabatic Berry phase, the Hamiltonian should be periodical i.e. $H(t + T) = H(t)$. If the transformation $U(t)$ has the same period as the Hamiltonian $H(t)$ and the initial state of the system is chosen as the eigenstate of H_0', it can be seen from equation (7) that the state should return to itself after a period T up to a Hill's phase $\alpha_n(T)$

$$\alpha_n(T) = E_n\int_0^T\eta(s)\,\mathrm{d}s. \tag{9}$$

The dynamic phase is the time integral of the instantaneous expectation value of the Hamiltonian $\delta_n = \int_0^T\langle\psi_n(t)|H(t)|\psi(t)\rangle\,\mathrm{d}t$. Following [8] the non-adiabatic Berry phase is obtained by using equations (3), (7) and (9):

$$\beta_n = \alpha_n - \delta_n = -\mathrm{i}\int_0^T\langle\phi_n|U^{-1}\frac{\partial U}{\partial t}|\phi_n\rangle\,\mathrm{d}t \tag{10}$$

which expresses that the non-adiabatic Berry phase is independent of the eigenvalue of the Hamiltonian, H_0, but depends on its eigenstate and corresponding unitary transformation U.

3. *N*-particle model and its solution

N particles moving in a two-dimensional parabolic potential, $\sum_{i=1}^{N}\frac{1}{2}\omega_0^2\vec{r}_i^{\,2}$, with repulsive interaction $\frac{\beta}{r^2}$, subjected to a magnetic field B along the z-axis. If the vector potential of the magnetic field varies with time, working in the symmetric gauge, $\vec{A}(t) = \frac{B(t)}{2}(-y, x, 0)$, the Hamiltonian of this system can be written as [6]

$$H = H_{space}(t) + H_{spin}(t) \tag{11}$$

$$H_{space}(t) = \sum_{i=1}^{N} \left(\frac{\vec{P}_i^2}{2m^*} + \frac{1}{2} m^* \omega^2(t) \vec{r}_i^2 + \frac{\omega_c}{2} l_i \right) + \sum_{i<j} \frac{\beta}{|\vec{r}_i - \vec{r}_j|^2} \tag{12}$$

$$H_{spin}(t) = -g^* \mu_B B(t) \sum_{i=1}^{N} s_{iZ} \tag{13}$$

where $\omega^2(t) = \omega_0^2 + \omega_c^2/4$ and $\omega_c = eB(t)/m^*c$ is the cyclotron's frequency. ω_0 can also vary with time, which means the harmonic potential can be time-dependent [9]. The momentum and position of the ith particle are given by the two-dimensional vectors $\vec{p}_i$ and $\vec{r}_i$, respectively. l_i is the z component of angular momentum. The Calogero-like models in high dimensions generally have a three-body potential present for solvability [1, 3, 4]. The above Hamiltonian is a special case where the three-body force is zero, and this is of some interest to the quantum dot community. It is also worth pointing out that the inverse-square interaction is scaling independent even when confined in a harmonic potential. This has important implications on the spectrum of a breathing model [10].

H_{spin} is the Zeeman energy and the eigenstates of $H_{spin}(t)$ are just the product of the spinors of individual particles. Due to $[H_{spin}(t), H(t)] = 0$, for simplicity, we neglect the influence of the $H_{spin}(t)$ and select the mass of the particles to be unity hereafter. In order to obtain the spacial state, the following consecutive transformations are introduced, which make the spacial state $|\psi(t)\rangle$ become $|\psi_1(t)\rangle$, $|\psi_2(t)\rangle$ and $|\psi_3(t)\rangle$, respectively:

$$|\psi(t)\rangle = U_1|\psi_1(t)\rangle = exp\left[-\mathrm{i}\lambda(t) \sum_{i=1}^{N} l_i \right] |\psi_1(t)\rangle \tag{14}$$

$$|\psi_1(t)\rangle = U_2|\psi_2(t)\rangle = \exp\left[\mathrm{i}C(t) \sum_{i=1}^{N} \vec{r}_i^2 \right] |\psi_2(t)\rangle \tag{15}$$

and

$$|\psi_2(t)\rangle = U_3|\psi_3(t)\rangle = \exp\left[\mathrm{i}D(t) \sum_{i=1}^{N} \vec{r}_i \cdot \vec{p}_i + \vec{p}_i \cdot \vec{r}_i \right] |\psi_3(t)\rangle \tag{16}$$

where $\lambda(t)$, $C(t)$ and $D(t)$ are real functions of time. These functions are chosen in such a way that the Hamiltonian $H_{space}(t)$, after these transformations, takes the HFB form according to the method shown in section 2.

It is easy to show that under these transformations the coordinate and momentum operators change in the following way:

$$U_1^{-1} x U_1 = x \cos\lambda + \mathrm{i} y \sin\lambda \qquad U_1^{-1} y U_1 = y \cos\lambda - \mathrm{i} x \sin\lambda$$

$$U_1^{-1} p_x U_1 = p_x \cos\lambda + \mathrm{i} p_y \sin\lambda \qquad U_1^{-1} p_y U_1 = p_y \cos\lambda - \mathrm{i} p_x \sin\lambda$$

$$U_1^{-1} \vec{r}_i^2 U_1 = \vec{r}_i^2 \qquad U_1^{-1} \frac{1}{|\vec{r}_i - \vec{r}_j|^2} U_1 = \frac{1}{|\vec{r}_i - \vec{r}_j|^2} \qquad U_1^{-1} \vec{p}_i^2 U_1 = \vec{p}_i^2 \tag{17}$$

$$U_2^{-1} \vec{r}_i U_2 = \vec{r}_i \qquad U_2^{-1} \vec{p}_i U_2 = \vec{p}_i + 2C(t)\vec{r}_i \tag{18}$$

$$U_3^{-1} \vec{r}_i U_3 = \vec{r}_i \exp[-2D(t)] \qquad U_3^{-1} \vec{p}_i U_3 = \vec{p}_i \exp[2D(t)]. \tag{19}$$

Using equations (3) and (17), we get the Hamiltonian $H_1(t)$ after transforming $H_{space}(t)$

with U_1:

$$H_1(t) = U_1^{-1} H_{space}(t) U_1 - \mathrm{i} U_1^{-1} \frac{\partial U_1}{\partial t} = \sum_{i=1}^{N} \left[\frac{\vec{P}_i^2}{2} + \frac{1}{2}\omega^2(t)\vec{r}_i^2 + \left(\frac{\omega_c}{2} - \frac{\mathrm{d}\lambda(t)}{\mathrm{d}t}\right) l_{iz} \right] + \sum_{i<j} \frac{\beta}{|\vec{r}_i - \vec{r}_j|^2}. \qquad (20)$$

Obviously, if $\lambda(t)$ is chosen to satisfy

$$\frac{\omega_c}{2} - \frac{\mathrm{d}\lambda(t)}{\mathrm{d}t} = 0 \qquad (21)$$

the l_i term disappears in equation (20). In other words, the magnetic field is removed by going to an appropriate rotating frame (using U_1). It must be noted that the rotational angular function $\lambda(t)$ has been determined by equation (21). Following the procedure of obtaining equation (20), the Hamiltonian $H_1(t)$ becomes $H_2(t)$, $H_3(t)$ respectively, after using U_2 and U_3 transformations whose generators are the monopole compression and dilatation operators:

$$H_2(t) = U_2^{-1} H_1(t) U_2 - \mathrm{i} U_2^{-1} \frac{\partial U_2}{\partial t} = \sum_{i=1}^{N} \left[\frac{\vec{P}_i^2}{2} + \left(\frac{\partial C(t)}{\partial t} + 2C^2(t) + \frac{1}{2}\omega_0^2(t) \right) \vec{r}_i^2 \right] + C(t) \sum_{i=1}^{N} (\vec{r}_i \cdot \vec{p}_i + \vec{p}_i \cdot \vec{r}_i) + \sum_{i<j} \frac{\beta}{|\vec{r}_i - \vec{r}_j|^2} \qquad (22)$$

and

$$\begin{aligned} H_3(t) &= U_3^{-1} H_2(t) U_3 - \mathrm{i} U_3^{-1} \frac{\partial U_3}{\partial t} \\ &= \exp[4D(t)] \sum_{i=1}^{N} \left[\frac{\vec{P}_i^2}{2} + \exp(-4D(t)) \left(\frac{\mathrm{d}C(t)}{\mathrm{d}t} + 2C^2(t) + \frac{1}{2}\omega_0^2(t) \right) \vec{r}_i^2 \right] \\ &\quad + \left(C(t) + \frac{\mathrm{d}D(t)}{\mathrm{d}t} \right) \sum_{i=1}^{N} (\vec{r}_i \cdot \vec{p}_i + \vec{p}_i \cdot \vec{r}_i) + \exp[4D(t)] \sum_{i<j} \frac{\beta}{|\vec{r}_i - \vec{r}_j|^2}. \end{aligned} \qquad (23)$$

In order to make $H_3(t)$ take a HFB form, $C(t)$, $D(t)$ are chosen to satisfy:

$$C(t) + \frac{\mathrm{d}D(t)}{\mathrm{d}t} = 0 \qquad (24)$$

$$\frac{\mathrm{d}C(t)}{\mathrm{d}t} + 2C^2(t) + \frac{1}{2}\omega_0^2(t) = \frac{\Omega^2}{2} \exp[8D(t)] \qquad (25)$$

where Ω is an arbitrary constant. Setting $\rho(t) = \exp[-2D(t)]$, equation (25) can be rewritten in brief form:

$$\frac{\mathrm{d}^2\rho}{\mathrm{d}^2 t} + \frac{1}{2}\omega^2(t) = \frac{\Omega^2}{\rho^3}. \qquad (26)$$

After those choices, H_3 is shown in the HFB form:

$$H_3(t) = \frac{1}{\rho^2(t)} \left[\sum_{i=1}^{N} \frac{\vec{P}_i^2}{2} + \frac{1}{2}\Omega^2 \sum_{i=1}^{N} \vec{r}_i^2 + \sum_{i<j} \frac{\beta}{|\vec{r}_i - \vec{r}_j|^2} \right] \equiv \frac{1}{\rho^2(t)} H_0 \qquad (27)$$

where H_0 is the Hamiltonian of the Calogero model in two-dimensional space and $\frac{1}{\rho^2(t)}$ is determined by the auxiliary equation (26). From the above, it is clear that the time-dependent Hamiltonian $H_{space}(t)$ with a term $\frac{1}{2}\omega_c(t) l_i$ can be transformed into a product form of a time-independent Calogero model and a time-dependent factor, which is the HFB form we want.

On account of equation (6), the solution of the Schrödinger equation of $H_3(t)$ is shown to be

$$|\psi_3(t)\rangle = \exp\left[-\mathrm{i}H_0\int_{t_0}^{t}\frac{1}{\rho^2(\tau)}\,\mathrm{d}\tau\right]|\psi_3(t_0)\rangle. \tag{28}$$

To obtain an explicit form of the solution, let us assume the initial state as: $|\psi_3(t_0)\rangle = \sum_n a_n|\phi_n\rangle$, where $|\phi_n\rangle$ is the eigenstate of the time-independent Calogero model's Hamiltonian H_0. We have

$$|\psi_3(t)\rangle = \sum_n a_n \exp\left[-\mathrm{i}E_n\int_{t_0}^{t}\frac{1}{\rho^2(\tau)}\,\mathrm{d}\tau\right]|\phi_n\rangle. \tag{29}$$

The exact solution of the Hamiltonian $H_{space}(t)$ can now be found by combining the above results. We finally obtain

$$|\psi(t)\rangle = U(t)|\psi_3(t)\rangle = \sum_n a_n \exp\left[-\mathrm{i}E_n\int_{t_0}^{t}\frac{1}{\rho^2(\tau)}\,\mathrm{d}\tau\right]U(t)|\phi_n\rangle = \sum_n a_n|\Psi_n(t)\rangle \tag{30}$$

where we have defined $U(t) = U_1U_2U_3$ and $|\Psi_n(t)\rangle = \exp[-\mathrm{i}E_n\int_{t_0}^{t}\frac{1}{\rho^2(\tau)}\,\mathrm{d}\tau]U(t)|\phi_n\rangle$.

It can be seen from equation (30) that if the eigenstates and eigenvalues of H_0 are known, so are the solutions of the time-dependent Hamiltonian $H_{space}(t)$. Unlike the one-dimensional Calogero model, the models in higher dimensions are only exactly solvable for the ground state and a class of excited states. Now we only consider the ground state and its eigenvalue of the Calogero model, which are given as follows [2]:

$$\begin{aligned}|\phi_0\rangle &= \prod_{i=1}^{N}\exp[-\Omega\vec{r}_i^{\,2}/2]\prod_{j>i=1}^{N}|\vec{r}_i-\vec{r}_j|^m\\ E_0 &= N[m(N-1)+1]\end{aligned} \tag{31}$$

where we have set $\beta = m(m-1)$. If $|\phi_0\rangle$ is chosen as the initial state $|\psi_3(t_0)\rangle$, the evolution of the state is written by using equation (7)

$$\begin{aligned}|\psi_0(t)\rangle = {}&\exp\left[-\mathrm{i}\int_{t_0}^{t}\frac{1}{2}\omega_c(\tau)\,\mathrm{d}\tau\sum_{i=1}^{N}l_i\right]\exp\left[-\mathrm{i}\frac{\dot{\rho}(t)}{2\rho(t)}\sum_{i=1}^{N}\vec{r}_i^{\,2}\right]\\ &\times\exp\left[-\mathrm{i}\frac{1}{2}\ln\rho(t)\sum_{i=1}^{N}(\vec{r}_i\cdot\vec{p}_i+\vec{p}_i\cdot\vec{r}_i)\right]\\ &\times\exp\left[-\mathrm{i}E_0\int_{t_0}^{t}\frac{1}{\rho^2(\tau)}\,\mathrm{d}\tau\right]\prod_{i=1}^{N}\exp[-\Omega\vec{r}_i^{\,2}/2]\prod_{j>i=1}^{N}|\vec{r}_i-\vec{r}_j|^m.\end{aligned} \tag{32}$$

It can be easily proved that the invariant operator of the Hamiltonian $H_{space}(t)$ is

$$I(t) = U(t)H_0U^{-1} = \frac{1}{2m^*}\sum_{i=1}^{N}(\rho\vec{p}_i-\dot{\rho}\vec{r}_i)^2 + \frac{1}{2\rho^2}\Omega_0^2\sum_{i=1}^{N}\vec{r}_i^{\,2} + \rho^2\sum_{i<j}\frac{\beta}{|\vec{r}_i-\vec{r}_j|^2} \tag{33}$$

and $|\Psi_n(t)\rangle$ is the eigenstate of $I(t)$ with eigenvalue E_n.

In order to give the non-adiabatic Berry phase, a periodical Hamiltonian $H_{space}(t+T) = H_{space}(t)$ is considered. According to equation (10) the non-adiabatic Berry phase of the system is

$$\beta(T) = -\sum_{i=1}^{N}\int_0^T\langle\phi_0(\tau)|\vec{r}_i^{\,2}|\phi_0(\tau)\rangle\oint\dot{\rho}\,\mathrm{d}\rho. \tag{34}$$

Equation (34) shows that due to the time-dependent magnetic field acting on the Calogero model, the non-adiabatic Berry phase appears in this system no matter whether there are trapped harmonic potentials or not.

4. Conclusion and acknowledgment

By performing the time-dependent unitary transformation, the two-dimensional Calogero model acted upon by a time-dependent external field can be transformed into a product of a time-independent Calogero model's Hamiltonian and a factor depending only on time. The invariant operator and the eigenstate of this system is presented formally. If the magnetic field is periodical in time, the adiabatic Berry phase always exists no matter whether there is harmonic potential or not.

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